Scaling of physical quantities in TRISTAN-MP

Tristan employs an unconventional system of electromagnetic units. It is best described as a hybrid between the Gaussian and the rationalized MKSA systems. Following Jackson (1975) the dynamical Maxwell's equations in an arbitrary electromagnetic system of units can be written as

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{c^2}{\alpha}\nabla \times \mathbf{B} - 4\pi k_1 \mathbf{J}, \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\alpha \nabla \times \mathbf{E}, \tag{2}$$

where k_1 and α are arbitrary constants, and c is the speed of light. For instance, for the Gaussian system $k_1 = 1$, $\alpha = c$, and for MKSA $k_1 = 1/4\pi\epsilon_0$ and $\alpha = 1$. In TRISTAN, the convention is $k_1 = 1/4\pi$ and $\alpha = 1$. The Maxwell's equations solved by the code then have the form:

$$\frac{\partial \mathbf{E}_T}{\partial t} = c^2 \nabla \times \mathbf{B}_T - \mathbf{J}_T, \tag{3}$$

$$\frac{\partial \mathbf{B}_T}{\partial t} = -\nabla \times \mathbf{E}_T, \tag{4}$$

where subscript T stands for TRISTAN system. Conversion to Gaussian field units (subscript G) is simple: $E_G = 4\pi E_T$, and $B_G = 4\pi c B_T$, where the speed of light is measured in cgs. In the TRISTAN system of electromagnetic units the Lorentz force has the form: $4\pi q_T(\mathbf{E_T} + \mathbf{v} \times \mathbf{B_T})$. Note, that $q_T = q_G$ as can be seen from the Poisson equation $\nabla \cdot \mathbf{E} = 4\pi k_1 \rho$.

Suppose we are to discretize the Maxwell's and Lorentz force equations on a grid with spacing Δx and timestep Δt, such that the Courant condition is satisfied: $c\Delta t/\Delta x = \hat{C}$. Here, $\hat{C} < 1$ is the Courant number (usually less than .5 for stability). Normalizing the field in terms of some fiducial field B_{T0}: $\hat{E}_T = E_T/cB_{T0}$, $\hat{B}_T = B_T/B_{T0}$, we can write the discretized equations as:

$$\Delta[\hat{\mathbf{E}}_T]_t = \hat{C}\Delta[\hat{\mathbf{B}}_T]_x - \frac{\mathbf{J}_T \Delta t}{cB_{T0}}, \tag{5}$$

$$\Delta[\hat{\mathbf{B}}_T]_t = -\hat{C}\Delta[\hat{\mathbf{E}}_T]_x, \tag{6}$$

$$\Delta[\hat{\mathbf{v}}]_t = 4\pi \frac{q_{mp}}{m_{mp}} \hat{C} B_{T0} \Delta t (\hat{\mathbf{E}}_T + \frac{\hat{\mathbf{v}}}{\hat{C}} \times \hat{\mathbf{B}}_T). \tag{7}$$

Here, $\Delta[...]_{t,x}$ is a shorthand for the differencing appropriate for either time advance or curl operation without the Δt or Δx multipliers, e.g. $\Delta[\hat{\mathbf{v}}]_t \equiv \hat{\mathbf{v}}^{n+1} - \hat{\mathbf{v}}^n$. Velocity is normalized in terms of $\Delta x/\Delta t$, and $q_{mp} = \hat{q} q_0$ and $m_{mp} = \hat{m} m_0$ stand for charge and mass of a macroparticle, normalized in terms of fiducial charge q_0 and mass m_0 (to be determined later).

In order to understand the scaling of quantities in TRISTAN we need to write out the expression for current density. Each macroparticle in the code is a cube of volume Δx^3, and the charge crossing the boundary between two cells in a timestep is calculated as a fraction of the volume of macroparticle that has moved through the boundary. Thus, $\mathbf{J}\Delta t = \hat{q}q_0/\Delta x^3\mathbf{v}\Delta t = \hat{q}q_0/\Delta x^3\Delta\hat{\mathbf{l}}\Delta x = \hat{q}\Delta\hat{\mathbf{l}}q_0/\Delta x^2$. Here, $\Delta\hat{\mathbf{l}}$ is the displacement of the particle in one timestep normalized in units of Δx. The units in the code are selected in such a way that the following relations hold:

$$4\pi\frac{q_0}{m_0}\hat{C}B_{T0}\Delta t = 1, \tag{8}$$

$$\frac{q_0}{\Delta x^2 c B_{T0}} = 1. \tag{9}$$

This way the equations in the code appear as if $\Delta t = \Delta x = c = q_0 = m_0 = B_{T0} = 1$. From relations (8-9) we can write $\omega_{c0}\Delta t = 1/\hat{C}$ and $\omega_{p0}\Delta t = 1$, where $\omega_{c0} \equiv 4\pi q_0 B_{T0}/m_0 = q_0 B_{G0}/(m_0 c)$ and $\omega_{p0}^2 = 4\pi q_0^2/(\Delta x^3 m_0)$ are fiducial cyclotron and plasma frequencies. By utilizing the Courant condition we can write an interesting constraint $\Delta x = 4\pi(q_0^2/m_0)(\hat{C}/c)^2$, which allows us to find the charge and mass scalings for the macroparticles. Suppose each macroparticle represents N electrons. Then, $\Delta x = 4\pi N(q_e^2/m_e)(\hat{C}/c)^2 = 4\pi N\hat{C}^2 r_e$, where $r_e = 2.8 \times 10^{-13}$cm is the classical electron radius, and each macroparticle "contains" $\hat{q}N = \hat{q}\frac{\Delta x}{4\pi\hat{C}^2 r_e}$ electrons or positrons. Ions can be selected by specifying $\hat{m} > 1$. Note, that independently of N, $\frac{q_0}{m_0} = \frac{\hat{q}}{\hat{m}}\frac{q_e}{m_e}$.

The field scaling is found from (8) by utilizing the charge/mass scalings above:

$$B_{T0} = \frac{1}{(4\pi)^2 N}\frac{m_e^2 c^3}{q_e^3\hat{C}^4} = \frac{1}{4\pi}\frac{m_e c}{q_e\hat{C}^2\Delta x} \tag{10}$$

In Gaussian units, the fiducial field is

$$B_{G0} = 4\pi c B_{T0} = \frac{1}{\hat{C}^2}\frac{q_e}{r_e\Delta x} \tag{11}$$

Therefore, in order to convert from the normalized code field quantities (denoted with a hat) to the Gaussian system, we use:

$$\mathbf{B}_G = \hat{\mathbf{B}}\frac{q_e}{\hat{C}^2 r_e\Delta x} = \frac{\hat{\mathbf{B}}}{\hat{C}^2}\frac{1.7\times 10^3\text{cm}}{\Delta x(\text{cm})}\text{Gauss} \tag{12}$$

$$\mathbf{E}_G = 4\pi c\hat{\mathbf{E}}B_{G0} = \frac{\hat{\mathbf{E}}}{\hat{C}^2}\frac{1.7\times 10^3\text{cm}}{\Delta x(\text{cm})}\text{statvolt/cm} \tag{13}$$

$$q_G = \hat{q}\left[\frac{\Delta x q_e}{4\pi\hat{C}^2 r_e}\right] = \frac{\hat{q}}{\hat{C}^2}136.4\Delta x(\text{cm})\text{statcoul} \tag{14}$$

The plasma and cyclotron frequencies can be determined from

$$\Omega_c = \frac{\hat{q}\hat{B}}{\hat{m}\hat{C}}\frac{1}{\Delta t} = \frac{\hat{q}\hat{B}}{\hat{m}\hat{C}^2}\left(\frac{c}{\Delta x}\right) \tag{15}$$

$$\omega_p = \sqrt{\frac{\hat{q}^2 N_{p/c}}{\hat{m}}}\frac{1}{\Delta t} = \sqrt{\frac{\hat{q}^2 N_{p/c}}{\hat{m}\hat{C}^2}}\left(\frac{c}{\Delta x}\right) \tag{16}$$

Here, $N_{p/c}$ is the number of macroparticles per cell. Let's go through the standard derivation of the plasma frequency to show how the TRISTAN system of units works. The normalized equation of motion for the particle that is solved by the code is: $\Delta[\hat{\mathbf{v}}]_t = \frac{\hat{q}}{\hat{m}}(\hat{\mathbf{E}}_T + \frac{\hat{\mathbf{v}}}{\hat{C}} \times \hat{\mathbf{B}}_T)$ Imagine charge separation in a plasma with density of $N_{p/c}$ macroparticles per cell. After displacement x of positive charges relative to the negative charges, the column density of charge accumulated is just $\sigma = \hat{q}N_{p/c}x$. Since $\nabla \cdot \mathbf{E}_T = \rho$, we get for the electric field between two planes of charge: $\hat{E}_T = \sigma$, after integrating over a Gaussian pillbox. Substituting this into the equation of motion (and ignoring the magnetic field) we get for the plasma frequency of oscillation: $\omega_p^2 = \frac{\hat{q}^2 N_{p/c}}{\hat{m}}$. Note, that since $\Delta t = 1$ in the code, this quantity can be interpreted as $\omega_p^2 \Delta t^2$. In order to get the relativistic quantities, factors of γ should be added accordingly. The scaling (12-14) is peculiar in the sense that scaling of all quantities depends on the physical size of the grid spacing. When we change the resolution of the simulation, we need to make sure that code field quantities are adjusted consistently. Once the grid spacing in physical units is chosen, the effective timestep of the simulation is set by the Courant number. It acts as if we are changing the speed of light in the code.